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MODE III DELAMINATION OF A VISCOELASTIC STRIP FROM A DISSIMILAR VISCOELASTIC HALF-PLANE

MICHAEL RYVKIN and LESLIE BANKS-SILLS
Department of Solid Mechanics, Materials and Structures, Faculty of Engineering,
Tel Aviv University, 69978 Ramat Aviv, Israel

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Abstract—The problem of a semi-infinite crack propagating steadily along the interface of a viscoelastic bimaterial composite is investigated. One of the constituents of the composite is a strip and the other is a dissimilar half-plane. The viscoelastic behavior of both materials is modeled as a standard solid. The crack is driven by an arbitrary traveling shear load applied to the crack faces, producing a state of antiplane strain (mode 3). The boundary value problem is reduced to a Wiener–Hopf equation and solved in closed form by means of Cauchy-type integrals. For the specific case of an exponentially decaying load, the expression for the stress intensity factor is derived and its behavior as a function of crack-tip speed for different material combinations is examined. For some limiting cases, the solution is seen to coincide with known results. The important problem of an elastic–viscoelastic composite is also considered.

1. INTRODUCTION

The important technical problem of interface crack propagation causing layer delamination is being studied by many authors. A rather complete study of the problem for elastic constituents may be found in Hutchinson and Suo (1991). Consideration of mixed mode interface fracture requires an investigation of not only modes 1 and 2, but also mode 3. These modes were studied by Jensen *et al.* (1990) for an elastic thin film which separates from a substrate composed of a different elastic material. The influence of mode 3 on the deformation pattern of the decohered thin film was emphasized in that investigation.

When viscoelastic effects are included in the analysis, it seems that only for mode 3 deformation may analytic results be obtained. Several papers have appeared in the literature for cracks propagating along the interface between two viscoelastic materials. Sills and Benveniste (1981) determined the stress intensity factor for a crack propagating steadily between two different viscoelastic half-planes, modeled as Maxwell materials, and driven by an exponentially decaying load applied to the crack faces. In a later paper, they (Banks-Sills and Benveniste, 1983) considered the same problem with the materials modeled as standard solids. Coussy (1987) considered a body of the same geometry but for another viscoelastic model. He studied transient effects for a suddenly appearing crack with a uniformly distributed load applied to the crack faces. Ryvkin and Banks-Sills (1993) obtained the solution for a crack propagating steadily along the interface of an inhomogeneous viscoelastic strip for a uniformly distributed load.

The objective of the present paper is to investigate steady, mode 3 crack propagation between two viscoelastic constituents in which one is a strip and the other is a half-plane. This problem may be associated with decohesion of a protective coating from a substrate or with any kind of thin film delamination. The constitutive equations for the two materials are taken as standard solids. From this model, a Maxwell material, as well as elastic behavior may be obtained.

As in previous investigations, the formulated boundary value problem is reduced to a Wiener–Hopf equation by means of the Fourier transform. This equation is viewed as a Riemann problem with either a continuous or discontinuous coefficient depending upon the range of the crack-tip velocity. A closed form solution is constructed employing Cauchy-type integrals which enable determination of an expression for the stress intensity factor. In Section 2, the mathematical formulation of the problem and the analysis are presented for a general loading and for the specific case of an exponentially decaying load applied to the crack faces. The behavior of the stress intensity factor for the specific loading and for different material combinations is examined analytically and numerically with graphical results exhibited in Section 3. Specific material combinations including elastic, elastic–viscoelastic, and viscoelastic composites are considered. For the degenerate case of a crack propagating parallel to the boundary of a homogeneous elastic half-plane, a simple formula for the stress intensity factor is obtained.

2. ANALYSIS

Consider mode 3 delamination of a viscoelastic strip from a dissimilar viscoelastic half-plane as a result of steady-state propagation of a semi-infinite crack (Fig. 1). For the antiplane deformation to be investigated in the fixed coordinate system (x_1, x_2, x_3) with $x_2 = 0$ being the interface, all components of the stress strain field will be functions only of the two coordinates x_1 and x_2. The only non-zero displacement $w^{(r)}(x_1, x_2, t)$ is in the x_3-direction. The values of the index $r = 1, 2$ denote the materials of the strip which occupy the region $0 \leqslant x_2 \leqslant h$ and the half-plane, respectively. The non-vanishing strains are given by

$$\varepsilon_{\omega 3}^{(r)} = \frac{1}{2} \frac{\partial w^{(r)}}{\partial x_\omega}, \tag{1}$$

with $\omega = 1, 2$.

The viscoelastic behavior of both the strip and half-plane is modeled as a standard solid. For each medium, the constitutive equations for non-zero stresses may be written as

$$\frac{\partial \sigma_{\omega 3}^{(r)}}{\partial t} + \beta_r \sigma_{\omega 3}^{(r)} = 2\mu_r \left(\frac{\partial \varepsilon_{\omega 3}^{(r)}}{\partial t} + \alpha_r \varepsilon_{\omega 3}^{(r)} \right), \tag{2}$$

where $1/\beta_r$ and $1/\alpha_r$ are the relaxation and creep times, respectively, μ_r are the instantaneous short time shear moduli. The only equation of motion not satisfied identically in each material is

$$\frac{\partial \sigma_{13}^{(r)}}{\partial x_1} + \frac{\partial \sigma_{23}^{(r)}}{\partial x_2} = \rho_r \frac{\partial^2 w^{(r)}}{\partial t^2}, \tag{3}$$

where ρ_r denotes density.

The boundary conditions generating the antiplane strain state consist of restricting the displacement on the outer strip boundary, namely

$$w^{(1)}(x_1, h, t) = 0, \tag{4}$$

and applying a traveling shear load to the crack faces

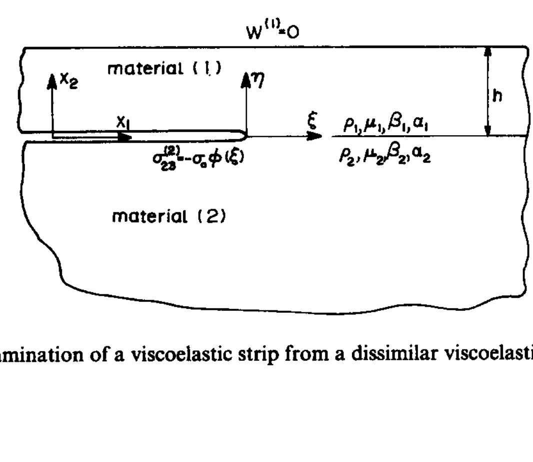

Fig. 1. Delamination of a viscoelastic strip from a dissimilar viscoelastic half-plane.

$$\sigma_{23}^{(r)}(x_1, 0, t) = -\sigma_0\phi(x_1 - vt), \quad x_1 - vt < 0. \tag{5}$$

In eqn (5), σ_0 is a constant with dimensions of stress, ϕ is some known function to be specified later, and v is the crack-tip velocity. The continuity conditions to be satisfied along the uncracked interface are

and

$$\left.\begin{aligned} \sigma_{23}^{(1)}(x_1, 0, t) &= \sigma_{23}^{(2)}(x_1, 0, t) \\ w^{(1)}(x_1, 0, t) &= w^{(2)}(x_1, 0, t) \end{aligned}\right\} x_1 - vt > 0. \tag{6}$$

To complete the formulation of the boundary value problem, the vanishing of the stresses far from the crack tip is assumed, so that

$$\sigma_{\omega 3}^{(r)}(x_1, x_2, t) \to 0, \quad x_1^2 + x_2^2 \to \infty. \tag{7}$$

As usual for problems with a uniformly moving crack tip, it is convenient to introduce a moving Galilean coordinate system (ξ, η) whose origin coincides with the crack tip and axes are parallel to the fixed axes (x_1, x_2), respectively (see Fig. 1). Thus

$$\xi = x_1 - vt, \quad \eta = x_2,$$

and, consequently

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = -v\frac{\partial}{\partial \xi}. \tag{8}$$

A Fourier transform in ξ is defined as

$$\bar{f}(s, \eta) = \int_{-\infty}^{\infty} f(\xi, \eta) \exp(\mathrm{i}s\xi)\,\mathrm{d}\xi, \tag{9}$$

where s is a complex variable and the inverse transform can be carried out along the real axis, defined as contour L. The boundary value problem in eqns (1)–(7) is rewritten in terms of the new coordinate system and (8). The Fourier transform is applied to these equations. The constitutive eqns (2) for each medium become

and

$$\left.\begin{aligned} \bar{\sigma}_{13}^{(r)} &= -\mathrm{i}s\bar{\mu}_r\bar{w}^{(r)} \\ \bar{\sigma}_{23}^{(r)} &= \bar{\mu}_r\frac{\mathrm{d}\bar{w}^{(r)}}{\mathrm{d}\eta} \end{aligned}\right\}, \tag{10}$$

where

$$\bar{\mu}_r = \frac{\mathrm{i}vs + \alpha_r}{\mathrm{i}vs + \beta_r}\mu_r \tag{11}$$

are complex shear moduli. The equations of motion (3) transform into

$$-\mathrm{i}s\bar{\sigma}_{13}^{(r)} + \frac{\mathrm{d}\bar{\sigma}_{23}^{(r)}}{\mathrm{d}\eta} = -v^2\rho_r s^2\bar{w}^{(r)}. \tag{12}$$

Substitution of the transformed stresses from eqn (10) into eqn (12) yields the ordinary differential equations

$$\frac{d^2\bar{w}^{(r)}}{d\eta^2} - \gamma_r^2 \bar{w}^{(r)} = 0, \quad r = 1, 2, \tag{13}$$

where

$$\gamma_r^2 = a_r^2 s^2 \frac{s - \mathrm{i}s_{r1}}{s - \mathrm{i}s_{r2}}, \tag{14}$$

$$s_{r1} = \frac{\alpha_r a_r^{*2}}{v a_r^2}, \quad s_{r2} = \frac{\alpha_r}{v}, \tag{15}$$

$$a_r^2 = 1 - v^2/c_r^2, \quad a_r^{*2} = 1 - v^2/c_r^{*2}. \tag{16}$$

The values $c_r^2 = \mu_r/\rho_r$ and $c_r^{*2} = \mu_r^*/\rho_r$ are the short and long time wave speeds, respectively, and $\mu_r^* = \mu_r \alpha_r/\beta_r$ are the long time shear moduli. Proceeding with the assumption that the branches of γ_r are chosen so that $\mathrm{Re}(\gamma_r) > 0$, it is possible to write the general solution of (13) for the strip in the form

$$\bar{w}^{(1)}(s, \eta) = A_1(s) \cosh[\gamma_1(\eta - h)] + B_1(s) \sinh[\gamma_1(\eta - h)] \tag{17}$$

and for the half-space in the form

$$\bar{w}^{(2)}(s, \eta) = A_2(s) \exp(-\gamma_2 \eta) + B_2(s) \exp(\gamma_2 \eta). \tag{18}$$

It may be immediately observed from the transformed boundary conditions (4) and (7) that $A_1 = A_2 = 0$.

Application of the Wiener–Hopf technique requires definition of "+" and "−" functions which are assumed to be analytic in upper, $\mathrm{Im}(s) \geqslant 0$, and lower, $\mathrm{Im}(s) \leqslant 0$, half-planes, respectively. These functions include the unknown tranform of the crack face displacement jump

$$W^-(s) = \int_{-\infty}^{0} [w^{(1)}(\xi, 0) - w^{(2)}(\xi, 0)] \exp(\mathrm{i}s\xi)\, d\xi, \tag{19}$$

the transform of the unknown continuous stresses along the uncracked interface

$$H^+(s) = \int_{0}^{\infty} \sigma_{23}^{(r)}(\xi, 0) \exp(\mathrm{i}s\xi)\, d\xi, \tag{20}$$

and the transform of the known crack face tractions

$$H^-(s) = -\sigma_0 \int_{-\infty}^{0} \phi(\xi) \exp(\mathrm{i}s\xi)\, d\xi. \tag{21}$$

By substituting (17) and (18) into the transformed continuity conditions (6) in the moving frame, one can find that

$$\left.\begin{aligned} B_1(s) &= -\bar{\mu}_2\gamma_2 \cosh^{-1}(\gamma_1 h)[\bar{\mu}_1\gamma_1 + \bar{\mu}_2\gamma_2 \tanh(\gamma_1 h)]^{-1} W^- \\ B_2(s) &= -\bar{\mu}_1\gamma_1[\bar{\mu}_1\gamma_1 + \bar{\mu}_2\gamma_2 \tanh(\gamma_1 h)]^{-1} W^- \end{aligned}\right\}. \tag{22}$$

Deriving next the expressions for the stress transform at the interface using relations (10) and (17), on the one hand, and (20) and (21), on the other hand, and eliminating from the resulting equation $B_1(s)$ by (22), leads to the Wiener–Hopf equation given by

$$H^{+}(s) = G(s)W^{-}(s) - H^{-}(s), \quad s \in L, \tag{23}$$

where

$$G(s) = -\left[\frac{\tanh(\gamma_1 h_1)}{\bar{\mu}_1 \gamma_1} + \frac{1}{\bar{\mu}_2 \gamma_2}\right]^{-1}. \tag{24}$$

It is worthwhile comparing the coefficient $G(s)$ with the corresponding coefficients for the problem of a crack propagating between two viscoelastic half-planes (Banks-Sills and Benveniste, 1983) and two bounded strips (Ryvkin and Banks-Sills, 1993). The coefficient $G(s)$ is found to be a mixture of the coefficients for the two problems mentioned, i.e. it contains the meromorphic "strip function" $\tanh(\gamma_1 h)/\bar{\mu}_1\gamma_1$ and "half plane function" $1/\bar{\mu}_2\gamma_2$ which has branch points.

Solution of (23) requires separation of $G(s)$ into two functions $G^{+}(s)$ and $G^{-}(s)$. This may be accomplished by considering the homogeneous equation

$$G^{+}(s) = G(s)G^{-}(s), \quad s \in L. \tag{25}$$

Assuming that

$$G(s) = G_1(s)G_2(s), \tag{26}$$

and solving two Riemann problems

$$G_i^{+}(s) = G_i(s)G_i^{-}(s), \quad s \in L, \tag{27}$$

where $i = 1, 2$, one can find the unknown functions $G^{\pm}(s)$ as

$$G^{\pm}(s) = G_1^{\pm}(s)G_2^{\pm}(s). \tag{28}$$

The functions $G_i(s)$ must be chosen so that it is possible to factor $G_1(s)$ by inspection and $G_2(s)$ using a Cauchy-type integral, given by

$$G_2^{\pm}(s) = \exp\left\{\frac{1}{2\pi i}\int_L \frac{\ln G_2(t)}{t-s}\,dt\right\}. \tag{29}$$

This representation is valid only if $G_2(s)$ satisfies on L the following conditions [see for example Gakhov (1966)]:

(a) $G_2(s)$ is Hölder continuous;
(b) $\lim_{\mathrm{Re}(s)\to\pm\infty} G_2(s) = 1$;
(c) the index of $G_2(s)$ on L is zero.

It will be shown later that the behavior of the function $G(s)$ and therefore the form of the functions $G_i(s)$ depends upon the relation between the crack-tip speed v and the long time wave speed of the half-plane material c_2^*. It may be noted that except for verification of condition (c) as will be pointed out, special treatment of $G(s)$ is not required when v is either less than or greater than c_1^*. Thus, two cases are distinguished, namely $0 \leqslant v \leqslant c_2^*$ and $c_2^* \leqslant v \leqslant c$ where $c = \min(c_1, c_2)$. It is convenient to begin with the case of large velocities.

Case (i) $c_2^* \leqslant v \leqslant c$

To factor the function $G(s)$ as in (26), it is useful to write

$$G(s) = -sQ^{-1}(s), \tag{30}$$

where

$$Q(s) = \frac{s \tanh(\gamma_1 h)}{\bar{\mu}_1 \gamma_1} + \frac{1}{\bar{\mu}_2 \hat{\gamma}_2} \tag{31}$$

and

$$\hat{\gamma}_2 = a_2 \operatorname{sgn}[\mathrm{Re}(s)](s - \mathrm{i}s_{21})^{1/2}(s - \mathrm{i}s_{22})^{-1/2} \tag{32}$$

and s_{21} and s_{22} are given in (15). In order to fulfill the assumption that the $\mathrm{Re}(\gamma_2) > 0$, the branch cuts for the square roots are chosen along the imaginary axis $(\mathrm{i}s_{21}, \mathrm{i}\infty)$ and $(\mathrm{i}s_{22}, \mathrm{i}\infty)$ [see Fig. 2(a)].

Following Kamisheva *et al.* (1982), the behavior of $G_2(s)$ is adjusted in order to satisfy condition (b) as

$$G_2(s) = \frac{Q(\infty)}{Q(s)} \coth \pi(hs + \mathrm{i}/4), \tag{33}$$

where

$$Q(\infty) = \lim_{\mathrm{Re}(s) \to \infty} Q(s),$$

so that

$$Q(\infty) = \frac{1}{\mu_1 a_1} + \frac{1}{\mu_2 a_2}. \tag{34}$$

It should be noted next that the value of s_{22} is always positive. The sign of s_{21} depends on the ratio v/c_2^*; in the case under consideration, $s_{21} < 0$. Therefore, for the branch cuts chosen, the ratio of the square roots in the expression for $\hat{\gamma}_2$ will have a jump for all points

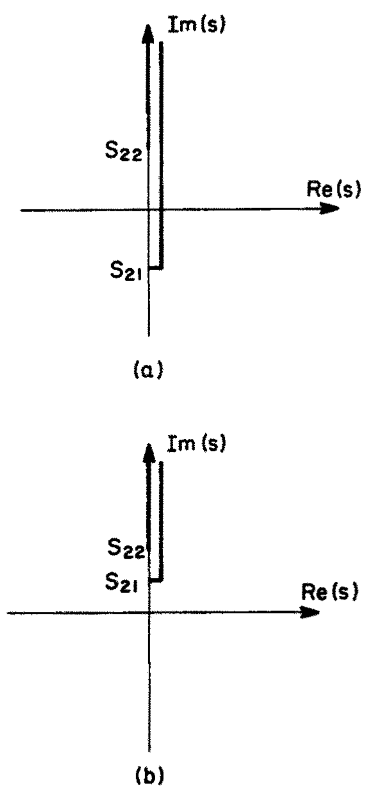

Fig. 2. Branch cuts in the s-plane for the functions $(s - s_{2j})^{1/2}$, $j = 1, 2$ for (a) $c_2^* \leqslant v \leqslant c$, and (b) $0 \leqslant v < c_2^*$.

s in the interval (is_{21}, is_{22}) and, in particular at $s = 0$. Because of the multiplicative function $\text{sgn}[\text{Re}(s)]$, $\hat{\gamma}_2$ itself, as well as $Q(s)$, will be continuous at these points, including $s = 0$. Taking this into account, it is clear that $G_2(s)$ is also continuous at $s = 0$ so that the continuity condition (a) is satisfied. Fulfillment of condition (b) is seen immediately from (31)–(33). To check satisfaction of condition (c), the techniques suggested by Ryvkin and Banks-Sills (1993) may be employed. If $c_1^* < c_2^*$, then verification of this condition is carried out numerically during the calculation of the stress intensity factor. If $c_1^* > c_2^*$, for $c_2^* \leqslant v \leqslant c_1^*$ condition (c) is seen to be valid analytically; for $c_1^* < v \leqslant c$, numerically.

In order to factor $G_1(s)$ which can be derived from (26) and (33), the identity

$$z \coth(\pi z) = \frac{\Gamma(1+iz)\Gamma(1-iz)}{\Gamma(\frac{1}{2}+iz)\Gamma(\frac{1}{2}-iz)}$$

is employed. The functions $G_1^{\pm}(s)$, analytic in the corresponding half-planes, are found to be

and

$$\left.\begin{aligned} G_1^+(s) &= \left(hs+\frac{i}{4}\right)\frac{\Gamma(\frac{3}{4}-ihs)}{\Gamma(\frac{5}{4}-ihs)} \\ G_1^-(s) &= -\frac{Q(\infty)}{hs}\frac{\Gamma(\frac{3}{4}+ihs)}{\Gamma(\frac{1}{4}+ihs)} \end{aligned}\right\}. \tag{35}$$

In reference to $G_1^-(s)$, the point $s = 0$ has been assumed to be situated in the upper half-plane.

Thus, the solution of the homogeneous equation (25) is given by (28), (29) and (31)–(35). Solution of the inhomogeneous equation (23) is now straightforward. Using the factorization in (25), eqn (23) may be rearranged as

$$\Phi^+(s) - \Phi^-(s) = g(s), \quad s \in L, \tag{36}$$

where

$$\Phi^+(s) = \frac{H^+(s)}{G^+(s)}, \quad \Phi^-(s) = \frac{W^-(s)}{G^-(s)}, \quad g(s) = -\frac{H^-(s)}{G^+(s)}. \tag{37}$$

Equation (36) is a routine problem of determination of a sectionally analytic function in accordance with a given jump. Proceeding with the assumption that the loading function $\phi(\xi)$ is sufficiently smooth and decreases so that $g(s)$ is Hölder continuous and tends to zero as $\text{Re}(s)$ becomes infinite, one can write

$$\Phi^{\pm}(s) = \frac{1}{2\pi i}\int_L \frac{g(t)}{t-s}\,dt. \tag{38}$$

In order to write the solution in the form shown in (38), the supplementary condition $\Phi^{\pm}(\infty) = 0$ is required. By considering the usual physical assumptions concerning the asymptotic behavior of the stress–strain field and the displacement jump near the crack tip, one may conclude that $H^+(s) = O(s^{-1/2})$ and $W^-(s) = O(s^{-3/2})$ as $|s| \to \infty$. Further, $G^+(s) = O(s^{1/2})$ and $G^-(s) = O(s^{-1/2})$ as $|s| \to \infty$, so that from (37) this condition is verified. Thus, the general solution for case (i) is constructed. By applying the inverse Fourier transform, it is possible to obtain all components of the stress–strain field.

To examine specific results, a particular applied load is chosen for consideration, namely

$$\phi(\xi) = \exp\left(\frac{\xi}{l}\right), \tag{39}$$

where l is a length parameter describing the loading decay rate. For such loading, the half Fourier transform integral (21) is given by

$$H^-(s) = \frac{i\sigma_0}{s - ib}, \tag{40}$$

where $b = 1/l$. The inhomogeneous equation (23) may be factored more simply than as given in (38). Indeed, employing (40), eqn (23) can be rewritten as

$$\frac{H^+(s)}{G^+(s)} + R^+(s) = \frac{W^-(s)}{G^-(s)} - R^-(s), \quad s \in L, \tag{41}$$

where

and

$$\left.\begin{aligned} R^+(s) &= \frac{i\sigma_0}{s-ib}\left[\frac{1}{G^+(s)} - \frac{1}{G^+(ib)}\right] \\ R^-(s) &= \frac{i\sigma_0}{(s-ib)}\frac{1}{G^+(ib)} \end{aligned}\right\}. \tag{42}$$

Functions on the right- and left-hand sides of (41) are analytic in upper and lower half-planes, respectively, and equal on contour L. Hence, they represent an analytical function in the entire s-plane. As mentioned previously, the behavior of the stresses and displacement jump near the crack tip, imply the behavior of $H^+(s)$ and $W^-(s)$ as $|s| \to \infty$. Employing the generalized Liouville theorem, one can find the unknown functions as

$$H^+(s) = -R^+(s)G^+(s) \tag{43}$$

and

$$W^-(s) = R^-(s)G^-(s). \tag{44}$$

In order to determine the stress intensity factor, which is of prime importance, the asymptotic behavior of the stresses $\sigma_{23}(\xi, 0)$ as $\xi \to 0^+$ must be evaluated. From (28), (29), (35), (42) and (43), the stress transform $H^+(s)$ is found to behave as

$$H^+(s) \sim -\frac{\sigma_0}{G^+(ib)}\exp\left(-\frac{i\pi}{4}\right)\left(\frac{h}{s}\right)^{1/2}, \quad |s| \to \infty. \tag{45}$$

Applying Abelian theorems (Noble, 1958), one obtains

$$\sigma_{23}^{(r)}(\xi, 0) \sim \sqrt{\frac{h}{\pi\xi}}\frac{i\sigma_0}{G^+(ib)}, \quad \xi \to 0^+. \tag{46}$$

In order to obtain a convenient expression for $G^+(ib)$, the conjugate property of the function $G_2(s)$ for real s, namely

$$\Psi(-\tau) = \bar{\Psi}(\tau), \quad \tau = \mathrm{Re}\,(s) \tag{47}$$

is employed. After some transformations including use of integral identities in which Gamma functions appear, the expression for $G^+(ib)$ is found to be

$$G^+(ib) = i\sqrt{bh}\exp(D), \tag{48}$$

where

$$D = \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\frac{\ln G_3(t)}{t-ib}\right]dt \tag{49}$$

and

$$G_3(t) = Q(\infty)Q(t)^{-1}. \tag{50}$$

Recall, $b = 1/l$.

The non-dimensional stress intensity factor is defined as

$$\hat{K}_{ve} = \frac{1}{K_0}\lim_{\xi\to 0^+}\sqrt{2\pi\xi}\sigma_{23}(\xi, 0) \tag{51}$$

where $K_0 = \sigma_0\sqrt{2l}$ is the stress intensity factor for a crack propagating steadily in an elastic inhomogeneous infinite plane under the same loading (39) as may be found in Atkinson (1977). From (46), (48) and (51)

$$\hat{K}_{ve} = \exp(-D), \tag{52}$$

which yields the final expression of the viscoelastic stress intensity factor for $c_2^* \leqslant v \leqslant c$. Note that the integral in (49) is evaluated numerically. The integrand in (49) is not singular; for $t \to \infty$ it decays algebraically as $O(t^{-2})$. Thus, there are no serious difficulties in performing the integration.

Case (ii) $0 \leqslant v \leqslant c_2^*$

For these values of v, s_{21} in (32) is positive causing the branch point to move into the upper half-plane, $\mathrm{Im}(s) > 0$. The branch cuts of the square roots in $\hat{\gamma}_2$ are chosen as in case (i) which may be seen in Fig. 2(b). Hence, $\hat{\gamma}_2$ and therefore, $G_2(s)$ in (33) are discontinuous for points on the imaginary axis in the neighborhood of $s = 0$; so that, condition (a) is not satisfied. Following Gakhov (1966) to obtain the solution of the Riemann problem (27) for $i = 2$ with a discontinuous coefficient, auxiliary functions $\omega^\pm(s)$ which are analytic in corresponding half-planes must be employed. For the problem under consideration, they may be defined by

$$\omega^+(s) = \frac{s^{1/2}}{(s+i/h)^{1/2}}, \quad \omega^-(s) = \frac{s^{1/2}}{(s-i/h)^{1/2}}. \tag{53}$$

The branch cuts for the square roots are taken as the parts of the imaginary axis $(0, i\infty), (-i/h, i\infty)$ and $(i/h, i\infty)$, respectively. Similar auxiliary functions were used by Walton (1982) for the problem of a crack propagating steadily in a homogeneous infinite viscoelastic body. The ratio of these functions

$$\Omega(s) = \frac{\omega^-(s)}{\omega^+(s)} = \frac{(hs+i)^{1/2}}{(hs-i)^{1/2}} \tag{54}$$

will be discontinuous for $s \in (-i/h, i/h)$. Applying this function to neutralize the discontinuity of the expression on the right-hand side of (33) leads to

$$G_2(s) = \Omega(s)\frac{Q(\infty)}{Q(s)}\coth\pi(hs+i/4). \tag{55}$$

By repeating the analysis carried out in the previous case, it is possible to show that $G_2(s)$ as chosen, satisfies conditions (a)–(c). Note that analytical verification of condition (c) may be carried out when $0 \leqslant v \leqslant c_1^*$. Thus, the solution of (27) for $i = 2$ can be found by (29). Corresponding to $G_2(s)$ as chosen, the function $G_1(s)$ may be factored by inspection. Using (26), (35) and (53)–(55), yields

and
$$\left.\begin{aligned} G_1^+(s) &= \left(hs+\frac{i}{4}\right)\frac{\Gamma(\frac{3}{4}-ihs)}{\Gamma(\frac{5}{4}-ihs)}w^+(s) \\ G_1^-(s) &= -\frac{Q(\infty)}{hs}\frac{\Gamma(\frac{3}{4}+ihs)}{\Gamma(\frac{1}{4}+ihs)}w^-(s) \end{aligned}\right\}. \tag{56}$$

From here, the derivation of the general solution for an arbitrary load given in (5) is the same as that for case (i). The solution is given in (37) and (38) with $G^+(s)$ and $G^-(s)$ taken from (28), (29), (55) and (56).

For the specific load given in (39), the factorization is again simplified as in (41). The expression for $G^+(ib)$ required in (46) for the stresses is obtained from the new expressions for G_1^+ and G_2^+. These are substituted into (28) and manipulated, taking note that

$$\int_0^\infty \mathrm{Im}\left[\frac{\ln \Omega(t)}{t-ib}\right]\mathrm{d}t = \left(\frac{bh+1}{bh}\right)^{1/2},$$

so that (48)–(50) remain valid. Therefore, the expression for the non-dimensional viscoelastic stress intensity factor in (52) is applicable for all possible values of the crack-tip velocity $0 \leqslant v \leqslant c$.

It may be noted that there is an alternate method for determining the stress intensity factor in (52). Following Ryvkin and Banks-Sills (1993), the problem of two bonded viscoelastic strips may be solved with an exponentially decaying load applied to the crack faces. Recall that their solution for this geometry was for a uniformly distributed applied load which may not be employed for a body constructed from a half-plane. At first glance, it may seem that letting one of the strip thicknesses approach infinity in the final expression for the stress intensity factor will lead to a solution for the problem at hand. But, some of the meromorphic functions appearing in this expression will now contain branch points. Since it is not clear if it is possible to employ this limiting process successfully, the authors preferred to solve the problem directly.

3. RESULTS

The obtained nondimensional stress intensity factor can be viewed as a function of the independent non-dimensional parameters of the problem, namely

$$\hat{K}_{ve} = \hat{K}_{ve}(v/c;\ h/l,\ \alpha_1/\beta_1,\ \beta_1 h/c_1,\ \alpha_1/\alpha_2,\ \beta_1/\beta_2,\ c_1/c_2, \mu_1/\mu_2), \tag{57}$$

where $c = \min(c_1, c_2)$. In this section, the behavior of the stress intensity factor as a function of the relative crack-tip velocity v/c is investigated for various material parameter combinations. Results are presented in graphical form in Figs 3–7. To better understand

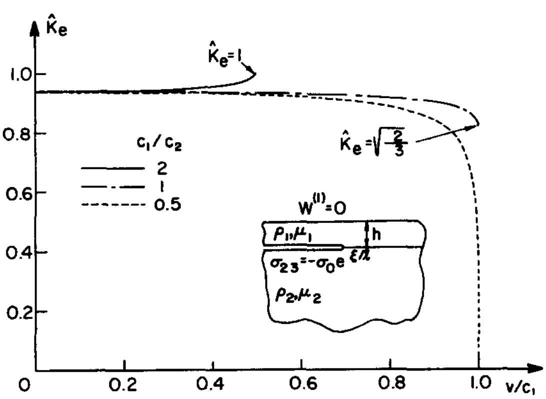

Fig. 3. Graph of the non-dimensional stress intensity factor $\hat{K}_e$ vs the non-dimensional crack-tip velocity v/c_1 for inhomogeneous elastic bodies with $h/l = 1$ and $\mu_1/\mu_2 = 2$.

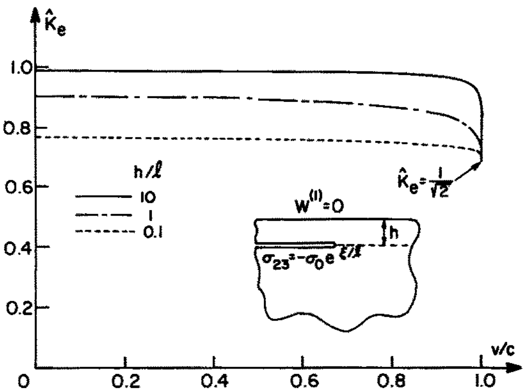

Fig. 4. Graph of the non-dimensional stress intensity factor $\hat{K}_e$ vs the non-dimensional crack-tip velocity v/c for a homogeneous elastic body for different ratios of the strip thickness h vs the loading length parameter l.

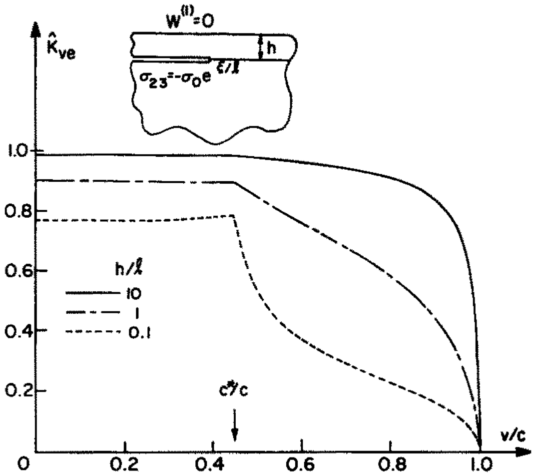

Fig. 5. Graph of the non-dimensional stress intensity factor $\hat{K}_{ve}$ vs the non-dimensional crack-tip velocity v/c for a homogeneous viscoelastic body with $\alpha/\beta = 0.2$ and $\beta h/c = 1$.

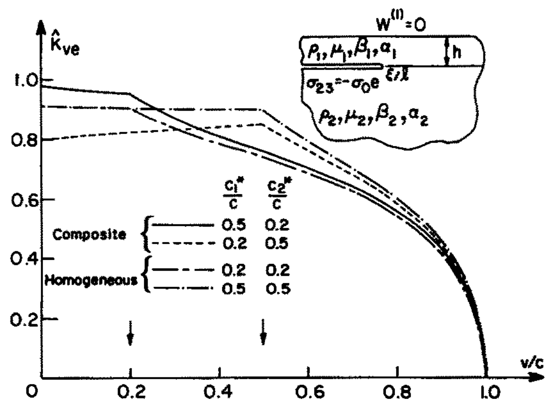

Fig. 6. Graph of the non-dimensional stress intensity factor $\hat{K}_{ve}$ vs the non-dimensional crack-tip velocity v/c for two viscoelastic composites and two viscoelastic homogeneous bodies with $c_1/c_2 = \mu_1/\mu_2 = \beta_1/\beta_2 = 1$, $\beta_1 h/c_1 = 1$, and $h/l = 1$. Other parameters are given in the text. The arrows along the abscissa denote critical values of c^*/c.

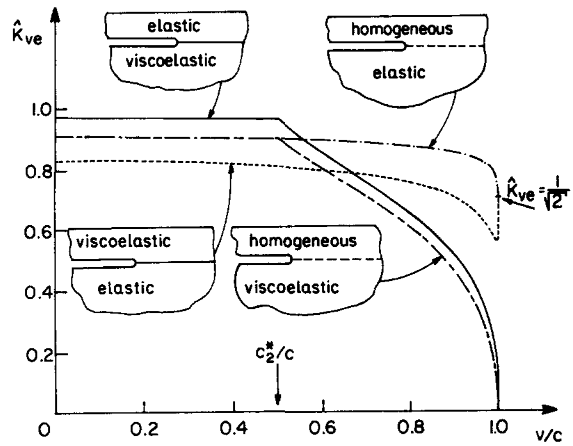

Fig. 7. Graph of the non-dimensional stress intensity factor $\hat{K}_{ve}$ vs the non-dimensional crack-tip velocity v/c for two elastic–viscoelastic composites with $c_1/c_2 = \mu_1/\mu_2 = 1$ and $h/l = 1$. The viscoelastic material parameters are $\alpha_r/\beta_r = 0.25$ and $\beta_r h/c_r = 1$. Also shown are two homogeneous bodies, one elastic and one viscoelastic.

the viscoelastic behavior, it is useful to consider first the degenerate case of two elastic materials. As may be seen from eqns (2) and (11), elimination of viscoelastic effects for both materials may be achieved by setting $\alpha_r = \beta_r$ $(r = 1,2)$. Substituting this into the solution in (52), one obtains the stress intensity factor for a crack propagating steadily between two elastic materials as

$$\hat{K}_e = \exp\left\{\int_0^\infty \frac{\ln Q_0(x)}{x^2+1}\,dx\right\}, \tag{58}$$

where

$$Q_0(x) = \left(\frac{\tanh(a_1 hx/l)}{a_1} + \frac{\mu_1}{a_2\mu_2}\right)\left(\frac{1}{a_1} + \frac{\mu_1}{a_2\mu_2}\right)^{-1}, \tag{59}$$

and a_i are defined in eqn (16). It may be observed that $0 < Q_0(x) < 1$, so that the integrand in (2) and thus, the integral itself, are negative. Hence, $\hat{K}_e < 1$ for any set of parameters. Note that intuitively it seems clear that the non-dimensional stress intensity factor obtained here is less than that determined for a crack propagating steadily between two dissimilar elastic half-planes. Recall that the solution to the latter problem is unity (Atkinson, 1977). The limiting value of the non-dimensional stress intensity factor $\hat{K}_{e0}$ for $v = 0$ is given by (58) and (59) with $a_1 = a_2 = 1$, so that

$$Q_0(x) = \left(\tanh\left(\frac{hx}{l}\right) + \frac{\mu_1}{\mu_2}\right)\left(1 + \frac{\mu_1}{\mu_2}\right)^{-1}. \tag{60}$$

For $v = c$, the value of the non-dimensional stress intensity factor $\hat{K}_{ec}$ is found to be

$$\hat{K}_{ec} = \begin{cases} 0 & \text{for } c_1 < c_2, \\ \sqrt{\dfrac{\mu_1/\mu_2}{1+\mu_1/\mu_2}} & \text{for } c_1 = c_2, \\ 1 & \text{for } c_1 > c_2. \end{cases} \tag{61}$$

Recall that the strip is material 1 and the half-plane is material 2. Indeed, this value depends upon the ratio of the shear wave speeds of the constituents. If the limiting value of the crack-tip velocity v is the strip wave speed c_1, then the non-dimensional stress intensity factor behaves as that found for a crack propagating steadily in an inhomogeneous elastic strip (Matczynski, 1974) which is zero. If, on the other hand, the limiting value is the half-plane wave speed c_2, then the non-dimensional stress intensity factor is unity, which is that found for a crack propagating steadily between two elastic half-planes (Atkinson, 1977). For $c_1 = c_2$, the non-dimensional stress intensity factor has neither behavior and tends to a constant between zero and unity. This constant depends upon the elastic modulus ratio μ_1/μ_2. Results obtained for the three wave speed combinations are depicted in Fig. 3. For all three cases $h/l = 1$ and $\mu_1/\mu_2 = 2$. From (58) and (60) it may be seen that the initial values of the non-dimensional stress intensity factor, when $v = 0$, are identical. For $v = c$, in accordance with (61), the $\hat{K}_e$ values are zero when $c_1 < c_2$, unity when $c_1 > c_2$ and $\sqrt{2/3}$ when $c_1 = c_2$.

It is interesting to consider now the important subcase of a crack propagating steadily a distance h from the boundary in a homogeneous elastic material. Setting $a_1 = a_2$ and $\mu_1 = \mu_2$ in (58) and (59) and carrying out the integration, one may obtain a closed form expression for the non-dimensional stress intensity factor, namely

$$\hat{K}_{eh} = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{a_1 h}{\pi l}+\frac{1}{2}\right)\exp\left\{\frac{a_1 h}{\pi l}\left(1-\ln\frac{a_1 h}{\pi l}\right)\right\}. \tag{62}$$

To the authors' knowledge, this result does not appear in the literature. Graphical results for three non-dimensional thicknesses $h/l = 0.1$, 1 and 10 are exhibited in Fig. 4. From (62) as well as Fig. 4, it may be seen that as the crack-tip speed v changes from zero to c, the non-dimensional stress intensity factor decays monotonically from its initial value $\hat{K}_{e0}$ to $1/\sqrt{2}$. This latter value may also be determined from (61) with $\mu_1/\mu_2 = 1$. As h/l increases, the non-dimensional stress intensity factor increases with its value approaching unity for $0 \leqslant v < c$. This may be expected since the limiting case $h \to \infty$ corresponds to the problem of a crack propagating steadily at the interface of an infinite homogeneous elastic plane.

Returning to consider viscoelastic materials, the limiting values of the non-dimensional stress intensity factor $\hat{K}_{v0}$ and $\hat{K}_{ve}$ for crack-tip velocities approaching zero and c, respectively, will be determined first. Following Ryvkin and Banks-Sills (1993), $\hat{K}_{v0}$ is determined by means of the correspondence principle. The elastic modulus ratio μ_1/μ_2 is replaced in the expressions for the elastic stress intensity factor in (58) and (60) by the ratio of the long time material moduli μ_1^*/μ_2^* so that

$$\hat{K}_{v0} = \hat{K}_{e0}(h/l, \mu_1^*/\mu_2^*). \tag{63}$$

In order to obtain the value $\hat{K}_{vc}$ as $v \to c$, it is possible to rewrite the integral in (49) in the form

$$D = \frac{1}{\pi}\int_0^\infty \frac{b\ln|G_3(t)| + t\arg G_3(t)}{t^2+b^2}\,\mathrm{d}t, \tag{64}$$

where $G_3(t)$ is given in (50). For $v \to c$ according to (14)–(16), (31), (32) and (34), the modulus of the function $G_3(t)$ becomes unbounded; so that the integral in (64) diverges, approaching plus infinity. Hence, from (52), $\hat{K}_{vc} = 0$.

To proceed, it is convenient to consider first the homogeneous case where the materials of the strip and half-plane are identical, namely $c_1 = c_2 = c$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, $\mu_1 = \mu_2 = \mu$. Three curves are exhibited in Fig. 5 for the non-dimensional parameter $h/l = 0.1$, 1 and 10. The remaining parameters in (57) are chosen to be $\beta h/c = 1$ and $\alpha/\beta = 0.2$ for all curves. Consequently, the relative long time wave speed in this homogeneous body is $c^*/c = \sqrt{0.2}$. It may be observed in the figure that the non-dimensional stress intensity factor does not change substantially for $0 \leqslant v \leqslant c^*$. As discussed, from the correspondence principle, the initial value $\hat{K}_{v0}$ coincides with the corresponding elastic one for the homogeneous case given in (62) with $a_1 = 1$. This value depends only upon the non-dimensional parameter h/l and as in the elastic case increases monotonically from $1/\sqrt{2}$ to unity when h/l varies from zero to infinity. For $c^* \leqslant v \leqslant c$, all three curves decrease to zero with the slope changing according to the value of h/l or alternatively, the non-dimensional parameter $A^* = c/\beta l$. It is worthwhile comparing these curves with those presented by Banks-Sills and Benveniste (1983) in their Fig. 3 for the non-dimensional stress intensity factor of a crack propagating steadily in a homogeneous finite viscoelastic plane subjected to the same applied loading as employed in this study. It may be noted that the behavior of the curves in their investigation for $c^* < v < c$ is controlled also by A^*. If this parameter is calculated here, it is found to assume the same values as in their study, namely 0.1, 1 and 10. Indeed, it has been observed in Banks-Sills and Benveniste (1983) and Atkinson and Popelar (1979) that typically for cracks propagating in standard solids, the behavior of the stress intensity factor changes at $v = c^*$.

Next, the general problem of dissimilar viscoelastic materials is addressed. It is useful first to obtain analytically the solution for the limiting case $h/l \to \infty$ for $0 < v < c^*$ where $c^* = \min(c_1^*, c_2^*)$. Writing $G_3(t)$ from (50) as a function of non-dimensional parameters and taking the limit, leads to

$$G_3(t) = Q^{-1}(\infty) \sum_{r=1}^{2} \bar{\mu}_r^{-1} a_r^{-1} (t - \mathrm{i}s_{r2})^{1/2} (t - \mathrm{i}s_{r1})^{-1/2}, \tag{65}$$

where the branch cuts corresponding to s_{11} and s_{12} are prescribed in a manner similar to s_{21} and s_{22} as in Fig. 2. The integral D is rewritten in the form

$$D = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\ln G_3(t)}{t - ib}\, \mathrm{d}t \tag{66}$$

and the integrand is analytically continued into the s-plane. It may be noted that for $0 < v < c^*$, all singularities and branch cuts of the integrand are situated in the upper half plane, namely $\mathrm{Im}(s) > 0$. Taking into account that the integrand behaves as $O(t^{-2})$ for large t and employing the Cauchy theorem yields $D = 0$. Consequently for crack-tip velocities in the range mentioned, the non-dimensional stress intensity factor $\hat{K}_{ve}$ is equal to unity, coinciding with that found by Banks-Sills and Benveniste (1983).

The behavior of two arbitrary viscoelastic composites is presented in Fig. 6 and compared to that of two corresponding homogeneous bodies. In one case (dotted line), the non-dimensional parameters are taken as $c_1/c_2 = \mu_1/\mu_2 = \beta_1/\beta_2 = h/l = \beta_1 h/c_1 = 1$, $\alpha_1/\beta_1 = 0.04$, $\alpha_1/\alpha_2 = 0.16$ and consequently the non-dimensional long time wave speed of the strip $c_1^*/c = 0.2$ is less than that of the half-plane, with $c_2^*/c = 0.5$. The second curve (solid line) depicts the case for which the materials of the strip and half-plane are exchanged. For comparison, two curves for corresponding homogeneous viscoelastic materials are exhibited. Since μ_1^*/μ_2^* is unity for each of these homogeneous bodies, their initial values. $\hat{K}_{v0}$, coincide as may be seen from (63). The initial value of $\hat{K}_{v0}$ for the composite bodies is determined from (58), (60) and (63). For the given thickness h/l, this value increases as the long time modulus ratio increases varying within the limits

$$\sqrt{\frac{h}{\pi l}} \frac{\Gamma\left(\dfrac{h}{\pi l} + \dfrac{1}{2}\right)}{\Gamma\left(\dfrac{h}{\pi l} + 1\right)} \leqslant \hat{K}_{v0} \leqslant 1. \tag{67}$$

The upper bound is expected since it represents the solution of a crack propagating steadily in a homogeneous viscoelastic body. Then for $v = 0$, the non-dimensional stress intensity factor can be either less than or greater than its corresponding homogeneous value as seen in Fig. 6. If, on the other hand, μ_1^*/μ_2^* is fixed, it may be easily shown that $\hat{K}_{v0}$ increases as a function of h/l and satisfies the relation

$$\sqrt{\frac{\mu_1^*/\mu_2^*}{1+\mu_1^*/\mu_2^*}} \leqslant \hat{K}_{v0} \leqslant 1. \tag{68}$$

Referring to Fig. 6, it may be observed that for each composite body, the difference between its curve and that of the homogeneous viscoelastic bodies decreases monotonically beginning from the initial value $\hat{K}_{v0}$. For those curves which approach one another as v increases, the material in the half-plane of the composite is the same as that of the homogeneous body. It is interesting to note further that the jump in the slope of the curves takes place only when the crack-tip velocity v reaches the long time wave speed c_2^* associated with the half-plane material; there is no visible change in the vicinity of the long time wave speed of the strip material c_1^*. Such behavior of the non-dimensional stress intensity factor of a crack propagating steadily between two standard solids, in which one is a strip and the other a half-space, contrasts with the results determined by Banks-Sills and Benveniste (1983) in which both constituents of the composite were half-planes. In their investigation, at each long time wave speed, a discontinuous slope was observed. On the other hand, it was observed by Ryvkin and Banks-Sills (1993) for a crack propagating steadily at the interface of two bonded strips that the slope of the stress intensity factor is continuous at both long time wave speeds. However, the value of $\hat{K}_{ve}$ changes precipitously at the long time wave speed of the material which is substantially thicker.

In Fig. 7 results are presented for two elastic–viscoelastic composites, a homogeneous elastic and a homogeneous viscoelastic body of the same constituents. The parameters for the viscoelastic media are taken to be the same as in Fig. 6 for the material whose relative long time wave speed is 0.5. The shear wave speed and modulus of the elastic media are the same as the short time wave speed and instantaneous shear modulus of the viscoelastic material. The solid curve corresponds to the case in which the upper material is elastic and the lower viscoelastic; for the dotted curve the materials are reversed. The curves for the homogeneous materials are reproduced from Figs 4 and 6. For the composite in which the half-plane is viscoelastic, the behavior of the non-dimensional stress intensity factor is essentially the same as that of the homogeneous viscoelastic body or the composite viscoelastic bodies in Fig. 6. It appears that the general behavior is not affected by the elastic strip. For the composite in which the half-plane is elastic and the strip is viscoelastic, the behavior for all v except in a small vicinity of the point $v = c$ (where $c = c_1 = c_2$) is similar to that of the homogeneous elastic material. This may be verified by comparing to the curve in this graph for the latter material or referring to Fig. 4. When the crack-tip velocity $v = c$, the value of $\hat{K}_{vc}$ is found to be $1/\sqrt{2}$ as for a homogeneous elastic body. The value of $\hat{K}_{ve}$ precipitously approaches this value. Further, the stress intensity factor for the composite in which the half-plane is elastic does not decrease monotonically to its value at $v = c$ as for the homogeneous elastic body. Relations (67) and (68) for $v = 0$ remain valid; for the elastic material μ_2^* is replaced by μ_2. It may be shown for a composite in which the half-plane is elastic and the strip viscoelastic but the instantaneous elastic moduli and short time wave speeds do not coincide (i.e. $\mu_1 \neq \mu_2$ and $c_1 \neq c_2$) that the limiting value of $\hat{K}_{vc}$ for $v = c$ is the same as that of an inhomogeneous elastic system given by (61).

4. CONCLUSIONS

The closed form expression of the stress intensity factor for a semi-infinite crack propagating steadily along the interface of a viscoelastic strip bonded to a dissimilar viscoelastic half-plane has been determined. The viscoelastic constitutive law was that of a standard solid. The behavior of the stress intensity factor as a function of the crack-tip velocity for different material parameter combinations was examined.

First some comments for a viscoelastic composite, i.e. one in which both constituents are standard solids. The limiting value of the stress intensity factor for a standing crack, i.e. when the crack-tip velocity is zero, depended strongly upon the ratio of the strip thickness vs the decay length of the applied load and the ratio of the long time shear moduli of the constituents. This limiting value is always less than that of the corresponding problem for the infinite homogeneous or inhomogeneous elastic plane; it may be recalled that this value is identical in both cases and does not depend upon the material properties. As the crack-tip velocity increases, the influence of the strip material decreases. The stress intensity factor tends to that of a homogeneous viscoelastic body of the same geometry possessing the properties associated with the material of the half-plane. Consequently, for this geometry and loading the viscoelastic behavior of the homogeneous body may be said to qualitatively describe that of the viscoelastic composite. This tendency of the thicker layer to dominate the viscoelastic composite fracture behavior was noted by Ryvkin and Banks-Sills (1993) for the inhomogeneous strip problem. Consideration of an elastic–viscoelastic composite in which the strip is elastic and the half-plane is viscoelastic confirms this phenomenon; namely, typical viscoelastic behavior emerges. On the other hand, when the strip is viscoelastic and the half-plane is elastic, the stress intensity factor tends to behave like that of the corresponding homogeneous elastic problem.

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